

# A note on the slightly supercritical Navier Stokes equations in the plane

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March 8, 2013

## Abstract

We produce a new proof of Tao's result on the slightly supercritical Navier Stokes equations. Our proof has the advantage that it works in the plane while Tao's proof works only in dimensions three and higher. We accomplish this by studying the problem as a system of differential inequalities on the  $L^2$  norms of the Littlewood Paley decomposition, along the lines of Pavlovic's proof of the Beale-Kato-Majda theorem.

## 1 Introduction

Our goal is to study the slightly supercritical Navier Stokes equations. We will work in an arbitrary dimension  $\mathbf{R}^d$ , although our results are only new when  $d = 2$ . The system we are dealing with then is

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -D^2 u + \nabla p, \quad (1.1)$$

together with the condition of incompressibility

$$\nabla \cdot u = 0, \quad (1.2)$$

and initial conditions

$$u(x, 0) = u_0(x, 0). \quad (1.3)$$

Here  $D$  represents a Fourier multiplier with symbol  $\frac{|\xi|^{\frac{d+2}{4}}}{(\log(2+|\xi|))^\gamma}$  with  $\gamma \leq \frac{1}{4}$ . The aim of this note is to prove:

**Theorem 1.1.** *The above system with smooth, compactly supported initial conditions has a global in time smooth solution.*

This theorem is only new in the case  $d = 2$ . For other dimensions, the results can be found in [Tao].

We remark briefly on the history of the problem. The slightly supercritical Navier Stokes equations were introduced by Tao in [Tao]. The goal was to quantify exactly how close standard techniques are to proving global solvability of the standard Navier Stokes equations. The conclusion was that in dimensions higher than two, these techniques do not get very close but get a little closer than is suggested by the standard notion of criticality. This generalized a similar observation about slightly supercritical wave equations with radial symmetry established in [Tao3]. Since Tao's paper [Tao], various authors have approached Tao's result in slightly different settings and with different points of view. (See e.g. [DKV], [CS].)

We remark apologetically that our own result is not physically well motivated. In the plane, the standard Navier Stokes equations are critical and so the slightly supercritical equation have less dissipation than the standard ones. It is curious however that Tao's argument seemed not to cover this case.

Our own approach is inspired by Remark 1.2 in Tao's paper [Tao]. There it is shown that if all the energy of a flow is concentrated in a single scale, one should expect a result up to  $\gamma = \frac{1}{2}$ . Of course, the energy need not be concentrated in a single scale. Tao's method of making the heuristic precise involves some Sobolev embedding type results that reach an endpoint at  $d = 2$  where they fail to work. We study the scales somewhat more precisely by estimating differential inequalities on the  $L^2$  norms of Littlewood Paley components of the flow. This is an adaptation of an argument of Pavlovic for the classic Beale-Kato-Majda theorem which may be found in her thesis. [Pav] We see that the reason that one cannot get an exponent stronger than  $\frac{1}{4}$  is that one can't rule out that the singularity happens merely by making successive scales pass supercriticality without their attaining the majority of the available energy.

**Acknowledgements:** The first author is partially supported by NSF grant DMS-1001607 and a fellowship from the Guggenheim foundation. He would like to thank Terry Tao for suggesting the problem. We would also like to thank Vlad Vicol for a helpful discussion.

## 2 Littlewood Paley trichotomy

We introduce a standard Littlewood-Paley decomposition. Let  $\phi : \mathbf{R}^+ \rightarrow \mathbf{R}$ , be a smooth function so that  $\phi(x) = 1$  for  $0 < x < 1$ , and  $\phi(x) = 0$  for  $x > 2$ . We then define  $P_0$  to be the multiplier operator whose symbol is  $\phi(|\xi|)$ . We define for  $j > 0$  that  $P_j$  is the multiplier operator whose symbol is  $\phi(\frac{|\xi|}{2^j}) - \phi(\frac{|\xi|}{2^{j-1}})$ . Note that the identity is the sum of

the  $P_j$ 's since the sum of the symbols telescopes. To simplify notation, we define  $P_k = 0$  when  $k$  is negative.

We are interested in the development of  $\|P_j u\|_{L^2}$  over time assuming that  $u$  satisfies the slightly supercritical Navier Stokes equations. Letting  $\omega = \nabla \times u$  be the vorticity, we see it is entirely equivalent to study the development of  $\|P_j \omega\|_{L^2} \sim 2^j \|P_j u\|_{L^2}$ . The norms are equivalent because of the divergence free property of  $u$ . Passing to the vorticity helps us since it eliminates the pressure which is secretly a non-linear term in  $u$ . We obtain the vorticity form of the slightly supercritical Navier Stokes equation:

$$\frac{\partial \omega}{\partial t} + (\nabla \times (u \cdot \nabla) u) + (u \cdot \nabla) \omega = -D^2 \omega.$$

Here the nonlinear part has broken into two terms, commonly referred to as the vortex stretching term, which vanishes when  $d = 2$  and the advection term. (For our purposes, both will be of equal strength. Applying a Littlewood-Paley component with  $\langle, \rangle$  denoting  $L^2$  inner product in space, we get

$$\left\langle \frac{\partial P_j \omega}{\partial t}, P_j \omega \right\rangle = -\langle P_j((\nabla \times (u \cdot \nabla) u) + (u \cdot \nabla) \omega), P_j \omega \rangle - \langle D^2 P_j \omega, P_j \omega \rangle. \quad (2.1)$$

Now by estimating the right hand side, we get bounds for the rate of change in time of  $\|P_j \omega\|_{L^2}$ .

**Lemma 2.1.** *With  $\omega$  smooth and satisfying the vorticity form of the slightly supercritical Navier Stokes equations, there is a universal constant  $C > 0$  so that we have the estimate*

$$\frac{d}{dt} \|P_j \omega\|_{L^2} \leq C \left( \left( \sum_{k \leq j+5} 2^{\frac{dk}{2}} \|P_k \omega\|_{L^2} \right) \sum_{\alpha=-5}^5 \|P_{j+\alpha} \omega\|_{L^2} + \sum_{k \geq j} 2^{\frac{dj}{2}} \sum_{\alpha=-5}^5 \|P_k \omega\|_{L^2} \|P_{k+\alpha} \omega\|_{L^2} \right) \quad (2.2)$$

$$- \frac{2^{\frac{(d+2)j}{2}}}{j^{2\gamma}} \|P_j \omega\|_{L^2}. \quad (2.3)$$

Lemma 2.1 is, by now, a completely standard application of the Littlewood Paley trichotomy see [Tao2]. All terms except for the dissipation term appear in Pavlovic's proof of Beale-Kato-Majda see [Pav].

The last term of the inequality 2.2, comes from the dissipation, namely the last term of equation 2.1. The remaining terms on the right hand side come from estimation of the nonlinear term, namely the first term on the right in 2.1.

The nonlinear term is estimated by breaking each appearance of  $u$  and  $\omega$  on the left hand side of the inner product into a sum of its Littlewood Paley components. We observe that purely on frequency support grounds, only three types of terms survive: the low-high terms, high-low terms, and high-high terms. (Since the frequency of the right hand side of the inner product in 2.1 is held fixed at around  $2^j$ , it is the only region of frequency support of the product on the left hand side that we need concern ourselves with.) We estimate each term, an integral of a product of three components by controlling one  $L^\infty$  norm and two  $L^2$  norms. We always use the  $L^\infty$  norm of the lowest frequency component. We replace the  $L^\infty$  norm by an  $L^2$  norm using Bernstein's inequality. The right-most term of the first line of 2.2 comes from the high-high part. (We've used none of the structure of the equation and in fact would get a better estimate from applying a div-curl lemma, but we don't need this.)

The other terms in the first line of the right hand side of 2.2 come from the low-high and high-low terms. Most of these are quite straightforward. But there is one type of term, coming from the advection part, which has to be controlled using a commutator inequality. Namely, we have to obtain estimates on terms of the form

$$\langle P_l u \cdot \nabla P_{j\pm 1} \omega, P_j \omega \rangle,$$

with  $l$  significantly smaller than  $j$ . The apparent difficulty here is that we have an extra derivative falling on the relatively high frequency part  $P_{j\pm 1} \omega$ . We observe however that if  $Q$  is a multiplier supported at frequency approximately  $2^j$  (whose multiplier has derivative bounded by  $2^{-j}$ ) then we can estimate

$$\langle [Q, P_l u \cdot \nabla] f, g \rangle \lesssim 2^l \|P_l u\|_{L^\infty} \|f\|_{L^2} \|g\|_{L^2}.$$

This commutator idea allows us to exploit the fact that  $P_l u \cdot \nabla$  is antisymmetric because of the div-free property of  $P_l u$ . To wit: If we let

$$Q = \sum_{\beta=-3}^3 P_{j+\beta}$$

we can calculate

$$\langle P_l u \cdot \nabla P_{j\pm 1} \omega, P_j \omega \rangle \tag{2.4}$$

$$= \langle P_l u \cdot \nabla P_{j\pm 1} Q \omega, P_j Q \omega \rangle \tag{2.5}$$

$$= \langle P_j P_{j\pm 1} P_l u \cdot \nabla Q \omega, Q \omega \rangle + \langle [P_{j\pm 1}, P_l u \cdot \nabla] Q \omega, P_j Q \omega \rangle \tag{2.6}$$

The second term is harmless because it is a commutator and the first term is harmless because  $P_j P_{j\pm 1} P_l u \cdot \nabla$  is almost skew adjoint - namely:

$$\langle P_j P_{j\pm 1} P_l u \cdot \nabla Q \omega, Q \omega \rangle = - \langle P_j P_{j\pm 1} P_l u \cdot \nabla Q \omega, Q \omega \rangle + \langle Q \omega, [P_{j\pm 1} P_j, P_l u \cdot \nabla] Q \omega \rangle.$$

Thus we can solve for the noncommutator term in terms of the commutator term.

We will now use Lemma 2.1 as a black box.

### 3 Main Argument

We are now ready to proceed with the main part of the argument. The idea very much follows remark 1.2 of [Tao]. Namely, we will produce a measure of progress towards blow-up and we will show that infinite progress towards blow-up requires infinite dissipation of energy thus leading to a contradiction.

We introduce some notation. We define

$$b_j(t) = 2^{\frac{(d+2)j}{2}} \|P_j u(t)\|_{L^2}.$$

Thus by Bernstein's inequality, we have that  $b_j(t)$  controls  $\|P_j \omega(t)\|_{L^\infty}$ . We let

$$c(t) = \sum_{j=0}^{\infty} b_j(t).$$

One may note that  $c(t)$  is the norm of  $u$  in the Besov space  $B_2^{\frac{d+2}{2},1}$ . This is the norm which implicitly controls growth in the Beale-Kato-Majda argument. Our argument will primarily consist in showing that  $c(t)$  remains bounded.

We rewrite Lemma 2.1 in terms of the quantities  $b_j(t)$ .

**Lemma 3.1.** *Under the hypotheses of Theorem 1.1, there is a universal constant  $C > 0$  so that we have the system of inequalities*

$$\frac{db_j}{dt} \leq C \left( \left( \sum_{k \leq j+5} b_k(t) \right) \sum_{\alpha=-5}^5 b_{j+\alpha}(t) + \sum_{k \geq j} 2^{d(j-k)} \sum_{\alpha=-5}^5 b_k(t) b_{k+\alpha}(t) \right) - \frac{2^{\frac{(d+2)j}{2}}}{j^{2\gamma}} b_j(t). \quad (3.1)$$

Let  $E$  denote  $\|u_0\|_{L^2}$ . Because energy is dissipating, we have for each  $j$  that

$$b_j(t) \leq E 2^{\frac{(d+2)j}{2}}. \quad (3.2)$$

We assume that  $c(t)$  goes to infinity. We denote by  $t_k$ , the first time at which  $c(t) \geq 2^k$ . We are interested in studying what happens between  $t_k$  and  $t_{k+1}$ .

Clearly for  $t_k < t < t_{k+1}$ , we have that by the definition of  $c(t)$  for any  $j$  the estimate

$$b_j(t) \leq 2^{k+1}. \quad (3.3)$$

However for  $j$  sufficiently large, the dissipation term begins to dominate the others. We define  $j_{k,u}$  to be the solution for  $j$  of the equation

$$2^k = \frac{2^{\frac{(d+2)j}{2}}}{j^\gamma}.$$

We prove the following barrier estimate. Roughly speaking, it says that when the Besov norm  $c(t)$  is small, we have that  $b_j(t)$  is not increasing for large  $j$  since dissipation dominates the nonlinear term. However in light of lemma 3.1, we see that shrinkage from dissipation is only in proportion to  $b_j(t)$  while growth from the nonlinear term can come from neighboring Littlewood Paley pieces. The barrier estimate comes from examining  $b_j(t)$  only when it is large compared to its neighbors.

**Lemma 3.2.** *For any  $t$  with  $t_k < t < t_{k+1}$ , we have the estimate*

$$b_j(t) \leq C 2^k 2^{\frac{j_{k,u}-j}{10}}, \quad (3.4)$$

with  $C$  a universal constant.

*Proof.* We observe that in light of the inequality 3.3, the barrier estimate 3.4 is only nontrivial for  $k > j_{k,u} + c$ , where  $c$  is a constant that depends only on  $C$  in 3.4. Now we assume that  $t$  is the last time up to which the estimate 3.4 holds and that  $l$  represents the scale at which the barrier estimate is breached. That is, we assume that the inequality 3.4 holds at time  $t$ , but  $b_l(t) = C 2^k 2^{\frac{l-k}{10}}$ , and  $\frac{db_l}{dt} \geq 0$ , so that it is possible that  $b_l$  will be larger at slightly later times. We now examine the inequality 3.1. We observe that the low-high and high-low terms satisfy the estimate

$$\left( \sum_{s \leq l+5} b_s(t) \right) \sum_{\alpha=-5}^5 b_{l+\alpha}(t) \lesssim 2^k b_l(t).$$

Similarly, we see that the high high terms are dominated by a geometric sum:

$$\sum_{s \geq l} 2^{d(l-s)} \sum_{\alpha=-5}^5 b_s(t) b_{s+\alpha}(t) \lesssim b_l(t)^2 \lesssim 2^k b_l(t).$$

However, by choosing  $C$  sufficiently large, we see that  $l$  is sufficiently larger than  $j_{k,u}$  so that  $\frac{2^{\frac{(d+2)l}{2}}}{l^\gamma}$  dominates  $2^k$ . Thus the negative term in inequality 3.1 dominates the positive term and we have shown  $\frac{db_l}{dt} < 0$ , a contradiction.  $\square$

Now in light of the estimates 3.2 and 3.2 we see that there is a universal constant  $c$  and a scale  $j_{k,d}$  so that

$$\left( \sum_{j=0}^{j_{k,d}} + \sum_{j=j_{k,u}+c} \right) b_k(t) \leq \frac{2^k}{10},$$

with

$$j_{k,u} - j_{k,d} + c \lesssim \log k,$$

as long as  $t_k < t < t_{k+1}$ . Thus there are at most  $\log k$  scales that can contribute to the growth of the Besov norm  $c(t)$  from  $2^k$  to  $2^{k+1}$ . We pigeonhole to find a scale with a lot of growth. It turns out that the worst case for us is when the growth occurs near scale  $j_{k,u}$ . We pick an  $\epsilon > 0$  sufficiently small to be determined later so that we are guaranteed that there is some integer  $s \geq -c$  so that we have, letting  $l = j_{k,u} - s$ , the quantity  $b_l(t)$  increasing by  $\gtrsim 2^{k-s\epsilon}$  in the time period between  $t_k$  and  $t_{k+1}$ . Moreover in light of the definition of  $c(t)$  and the inequalities 3.3 and 3.4, we can control the right hand side of the inequality 3.1 to obtain that for  $t_k < t < t_{k+1}$ , we have

$$\frac{db_l}{dt} \lesssim 2^{2k},$$

so that the increase of  $b_l(t)$  takes place over a time period of length  $\gtrsim 2^{-k-s\epsilon}$ . It turns out that this guarantees us enough dissipation to reach our conclusion.

Recall that from conservation of energy we have that the dissipation is bounded:

$$\int_0^\infty \langle Du, Du \rangle dt \tag{3.5}$$

$$\sim \sum_{j=0}^\infty \int_0^\infty \frac{2^{\frac{(d+2)j}{2}}}{j^{2\gamma}} \|P_j u(t)\|_{L^2}^2 dt \tag{3.6}$$

$$\sim \sum_{j=0}^\infty \int_0^\infty \frac{2^{-\frac{(d+2)j}{2}}}{j^{2\gamma}} b_j(t)^2 dt \tag{3.7}$$

$$\lesssim E^2 \tag{3.8}$$

Now we use the fact that  $b_l(t)$  is  $\gtrsim 2^{k-s\epsilon}$  for a time period of  $\gtrsim \frac{1}{2^{k+s\epsilon}}$  to obtain a lower bound on the  $l$ th term in the third line of 3.5.

We calculate that

$$\frac{2^{-\frac{(d+2)l}{2}}}{l^{2\gamma}} = \frac{2^{-\frac{(d+2)l}{2}} l^{2\gamma}}{l^{4\gamma}} \sim \frac{2^{-k} 2^{\frac{(d+2)s}{2}}}{k^{4\gamma}},$$

where in the final equality we used the definition of  $j_{k,u}$  and the fact that  $l \sim k$ . Thus estimating the  $l$ th term of the third line of 3.5, we see that we get  $\gtrsim \frac{2^{\frac{(d+2)s}{2}}}{k^{4\gamma} 2^{3\epsilon s}}$  dissipation

between  $t_k$  and  $t_{k+1}$ . Choosing  $\epsilon < \frac{d+2}{6}$ , we get that the total amount of dissipation between  $t_k$  and  $t_{k+1}$  is  $\gtrsim \frac{1}{k^{4\gamma}}$ . The sum of these quantities diverges as long as  $\gamma \leq \frac{1}{4}$  and so we reach a contradiction. What we have shown is that the Besov norm  $c(t)$  is uniformly bounded.

Now all that remains is the relatively simple and standard task of showing that control on the Besov norm guarantees that an initial solution remains smooth. For any  $\mu > 0$ , there is a constant  $F$  so that

$$b_j(0) \leq F2^{-\mu j}.$$

Now we observe one more barrier estimate.

**Lemma 3.3.** *Let  $c(t) < M$  for all time  $t$ . There is a universal constant  $K$  so that for all  $j$*

$$b_j(t) \leq e^{KMt} F2^{-\mu j}. \quad (3.9)$$

*Proof.* As before, we assume that the inequality 3.9 holds until time  $t$  and  $b_j(t) = e^{KMt} F2^{-\mu j}$ . (Note that just the definition of  $c(t)$  guarantees us that this quantity is at most  $M$ .) Then we just examine the positive terms in the inequality 3.1. We see immediately that

$$\frac{db_j}{dt} \lesssim Mb_j(t) + b_j(t)^2 \lesssim Mb_j(t).$$

Thus by Gronwall's inequality, the barrier can't be broken.

□

In light of the fact that  $c(t)$  is uniformly bounded and in light of Lemma 3.3, we have proven Theorem 1.1.

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